

Linear algebra section 4

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Definition 1. Let V be a vector space over $\mathbb{R}$. A subset $W \subseteq V$ is called a subspace of V if it satisfies the following conditions:

1. $\mathbf{0} \in W$ (the zero vector of V is in W),
2. If $u, v \in W$, then $u + v \in W$ (closure under addition),
3. If $u \in W$ and $c \in \mathbb{F}$, then $cu \in W$ (closure under scalar multiplication).

Fact: Let T be a linear transformation, then $\text{image}(T)$, $\ker(T)$ form a subspace.

Problem 1. Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 1 & 1 & 2 & -1 \\ 2 & 5 & -1 & 2 & 4 \end{pmatrix},$$

a 3×5 matrix. Consider the linear transformation

$$T_A : \mathbb{R}^5 \rightarrow \mathbb{R}^3, \quad T_A(x) = Ax.$$

Find the kernel of T_A and a basis for $\ker(A)$.

hint The reduced row echelon form of A is

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 2. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \\ 3 & 6 \end{pmatrix}.$$

1. Compute the reduced row echelon form $\text{RREF}(A)$.
2. Find the image (column space) of A , viewed as a subspace of $\mathbb{R}^3$.
3. Find the image of $\text{RREF}(A)$.
4. Compare $\text{Im}(A)$ and $\text{Im}(\text{RREF}(A))$. Are they the same subspace of $\mathbb{R}^3$?

Very important point: You can find basis of kernel using rref but not the basis of image using rref!! Please do not make this mistake.

Problem 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Find vectors in $\mathbb{R}^3$ that span the image of A . What is the smallest number of vectors needed to span the image of A ?

Problem 4. Let $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be an injective linear map (that means two elements in $\mathbb{R}^m$ don't map to the same vector in $\mathbb{R}^n$.)

1. How can you compare m and n
2. What is an obvious choice of basis for $\text{image}(T)$?

Problem 5. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ determined by taking reflection of a vector v about the plane $1000x + 90y + 39z = 0$. Express the basis of its kernel and image. (You don't always need to find a matrix of a linear transformation to find its image and kernel, try to think geometrically about what is happening.)