

Linear algebra section 5

Yashi

January 2026

The goal of this section is to get comfortable with the notion of a vector space and subspace and getting comfortable with "non-obvious" looking vector spaces.

Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 1 & 3 \\ 5 & 8 \end{pmatrix}$$

with respect to the standard basis vectors. Since T is invertible, $\vec{f}_1 = T\vec{e}_1$ and $\vec{f}_2 = T\vec{e}_2$ also form a basis. Let $\mathcal{B}_1 = \{\vec{e}_1, \vec{e}_2\}$ and $\mathcal{B}_2 = \{\vec{f}_1, \vec{f}_2\}$, then the change of basis matrix $S_{\mathcal{B}_1 \rightarrow \mathcal{B}_2}$ is $A = \begin{pmatrix} 1 & 3 \\ 5 & 8 \end{pmatrix}$. Obviously, the change of basis matrix $S_{\mathcal{B}_2 \rightarrow \mathcal{B}_1} = A^{-1}$.

Now note that we can write $\vec{f}_1 = \vec{e}_1 + 5\vec{e}_2$, $\vec{f}_2 = 3\vec{e}_1 + 8\vec{e}_2$. It will be helpful to keep this basic point of view as well in mind from now on to deal with vector spaces that are isomorphic to $\mathbb{R}^n$ but on the face are not $\mathbb{R}^n$. (More on isomorphisms later!)

A finite vector space over $\mathbb{R}$ is always of the form $\mathbb{R}^n$, but sometimes it is $\mathbb{R}^n$ in disguise. The following problem will illustrate why.

Problem 1. *Show that the space of polynomials in degree less than equal to 2 and coefficients in $\mathbb{R}$ is a vector space.*

1. *Find its dimension and give two different set basis.*
2. *Find the change of basis matrix from the first matrix to the other.*

Problem 2. *(Work in groups of 3-4)*

Find whether the following spaces form a vector space. If yes, find their dimensions:

1. *The space of all 2×2 matrices.*
2. *The space of **non-invertible** 2×2 matrices.*
3. *The space of **invertible** 2×2 matrices.*
4. *The space of all 2×2 matrices S such that*

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Problem 3. Consider the following two different basis of the set of 2×2 matrices:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Find the change of basis matrix $S_{\mathcal{B}_1 \rightarrow \mathcal{B}_2}$ that takes you from $\mathcal{B}_1$ to $\mathcal{B}_2$?

There are natural vector spaces of infinite dimension as well!! See below:

Problem 4. (Challenge) Consider the linear space V of all infinite sequences of real numbers. We define the subset W of V consisting of all sequences $(x_0, x_1, x_2, \dots)$ such that

$$x_{n+2} = x_{n+1} + 6x_n \quad \text{for all } n \geq 0.$$

- (a) Show that W is a subspace of V .
- (b) Determine the dimension of W .
- (c) Does W contain any geometric sequences of the form $(1, c, c^2, c^3, \dots)$ for some constant c ? Find all such sequences in W .
- (d) What if the condition for the sequence were changed by $x_{n+2} = x_{n+1} - 6x_n$