

Linear algebra section 7

Yashi

January 2026

Problem 1. Consider two different basis of $\mathbb{R}^2$. $B_1 = \{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. Find the change of basis matrix $S_{B_1 \rightarrow B_2}$. Verify $S_{B_1 \rightarrow B_2}[\vec{a}_1, \vec{a}_2] = [\vec{b}_1, \vec{b}_2]$ and $[\vec{b}_1, \vec{b}_2] = [\vec{a}_1, \vec{a}_2]S_{B_1 \rightarrow B_2}$.

Problem 2. The equation $x_1 + x_2 + x_3 = 0$ defines a plane V in $\mathbb{R}^3$. In this plane, consider the two bases

$$\mathcal{A} = (\vec{a}_1, \vec{a}_2) = \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

and

$$\mathcal{B} = (\vec{b}_1, \vec{b}_2) = \left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right).$$

- (a) Find the change of basis matrix S from $\mathcal{B}$ to $\mathcal{A}$.
- (b) Verify that the matrix S in part (a) satisfies the equation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} S.$$

Explain why only one of the identities involving $S_{\mathcal{A} \rightarrow \mathcal{B}}$ holds in this case.

Problem 3. Let V be the space of all upper triangular 2×2 matrices. Consider the linear transformation

$$T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = aI_2 + bQ + cQ^2$$

from V to V , where

$$Q = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

(a) Find the matrix A of T with respect to the basis

$$\mathcal{A} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

(b) Find bases of the image and kernel of T , and thus determine the rank of T .

(c) Find the matrix of T with respect to the basis

$$\mathcal{B} = (I_2, Q, Q^2).$$

. Do it using two methods: one using change of basis matrix and one direct computations and compare your answers!

Problem 4. Consider the set of complex number $\mathbb{C}$ as a vector space over $\mathbb{R}$. Can you construct an isomorphism between the vector space $P_1(\mathbb{R})$ and $\mathbb{C}$?